Generalized Chapman-Kolmogorov equation: A unifying approach to the description of anomalous transport in external fields

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The generalized Chapman-Kolmogorov equation [V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, J. Stat. Phys. **9**, 45 (1973)] is discussed. It is demonstrated that this equation unifies recently proposed kinetic equations of fractional order that describe anomalous transport in external fields, as well as continuous time random walks. The conditions under which the individual models can be established are discussed.

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I. INTRODUCTION

The Chapman-Kolmogorov equation describes the probabilistic transition from a given state of a stochastic process to another state, via all the possible intermediates [1–5]. Its formulation dates back to Bachelier's treatises of stock market speculation [6], Smoluchowski's work on colloidal particles [7], Chapman's studies of the diffusion of grains in a nonuniform fluid [8], and Kolmogorov's probability theoretical investigation [9]. Consequently, the Chapman-Kolmogorov equation is often credited to Bachelier or Smoluchowski. It is called a chain equation by Montroll and West [10]. The Chapman-Kolmogorov equation is necessarily fulfilled by a Markov process [11].

In the following, we employ Chandrasekhar's notation [12] according to which the Chapman-Kolmogorov equation takes on the form

$$W(x,v,t+\Delta t)=\int_{-\infty}^{\infty}d(\Delta x)\int_{-\infty}^{\infty}d(\Delta v)W(x-\Delta x,v-\Delta v,t)\times\Psi(x-\Delta x,v-\Delta v;\Delta x,\Delta v) \tag{1}$$

in phase (position-velocity) space. Equation (1) describes the temporal evolution of the probability density function (pdf) $W(x,v,t)$ through the incremental transition from $W(x-\Delta x,v-\Delta v,t)$ to $W(x,v,t+\Delta t)$ during the average time step Δt. The transfer kernel in Eq. (1) is thereby given through [12]

$$\Psi(x-\Delta x,v-\Delta v;\Delta x,\Delta v)=\psi(v-\Delta v;\Delta v)\,\delta(\Delta x-v\Delta t). \tag{2}$$

The kernel Ψ and its factorized counterpart ψ describe the distribution of transitions with the velocity increment Δv for the field variables v and x where the position increment is connected with the mean time step Δt through $\Delta x=v\Delta t$.

Brownian motion [13], as described through the Chapman-Kolmogorov equation (1) is subject to the central limit theorem and therefore exhibits the linear time dependence

$$\langle x(t)^2\rangle=2Kt \tag{3}$$

of the force-free mean squared displacement, in one dimension. The diffusion constant K is of dimension $[K]=\mathrm{cm}^2\,\mathrm{sec}^{-1}$ [14,15].

There exists a growing number of systems for which deviations from the classical pattern (3) are reported [5,16–21]. In the following, we concentrate on such systems exhibiting force-free anomalous diffusion defined through

$$\langle x(t)^2\rangle=2K_\alpha^* t^\alpha,\quad \alpha\neq 1, \tag{4}$$

where α is called the anomalous diffusion exponent. The generalized diffusion constant K_α^* has dimension $[K_\alpha^*]=\mathrm{cm}^2\,\mathrm{sec}^{-\alpha}$. According to the value of α, one distinguishes slow or subdiffusion $(0<\alpha<1)$, and enhanced or superdiffusion $(\alpha>1)$, which includes the intermediate (subballistic, $1<\alpha<2$) range.

The description of anomalous diffusion in the absence and presence of an external force field has received considerable attention, and among the applied approaches one finds fractional Brownian motion [22], generalized master equations [23], continuous time random walk theory [24], generalized Langevin equations [25], or generalized q thermostatistics [26], just to name a few. In what follows, we concentrate on systems that are nonlocal in time, i.e., that display self-similar memory effects, that are linear, and that equilibrate toward the thermal Gibbs-Boltzmann equilibrium

$$W_{\mathrm{st}}=N\exp(-\beta E), \tag{5}$$

where N is a normalization factor, $\beta\equiv(k_BT)^{-1}$ is the Boltzmann factor, and $E=V(x)+T(v)$ denotes the energy, with $T(v)=(m/2)v^2$ representing the kinetic energy and $V(x)=$

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$-\int^{x}F(x')dx'$, the external potential. It has been argued thatsuch systems can be conveniently modeled in terms of fractional kinetic equations [27–40]. We also discuss the relation of the fractional approach to the continuous time random walk scheme (Lévy walks, Lévy flights, etc.), and the generalized master equation.

Our considerations are based on the generalized Chapman-Kolmogorov equation [39]

$$W(x,v,t)=\int_0^t dt'\int_{-\infty}^{\infty}d(\Delta x)\int_{-\infty}^{\infty}d(\Delta v)W(x-\Delta x,v-\Delta v,t')\Psi(x-\Delta x,v-\Delta v;\Delta x,\Delta v)w(t-t')+\phi(t)W_0(x,v), \tag{6}$$

which is the phase-space generalization of the force-free generalized master equation established for the continuous time random walk in Ref. [41]. Accordingly, in Eq. (6), the quantity $w(t)$ is the waiting time pdf that controls the time elapsing between any two successive jump events, and $W_0(x,v)\equiv\lim_{t\to0+}W(x,v,t)$ is the initial condition that persists with a temporally decaying amplitude $\phi(t)\equiv1-\int_0^t dt'w(t')$ designating the probability distribution of having encountered no jump event up to time t [42]. Through the choice of special forms for the waiting time pdf $w(t)$ and the transfer kernel $\Psi(x-\Delta x,v-\Delta v;\Delta x,\Delta v)$, we recover some models discussed in literature and are able to dwell on their relation.

After a primer on the classical Chapman-Kolmogorov equation (1) and its related kinetic equations of physical stochastic processes, we discuss the connection to some recently reported fractional models. Here and in what follows, we restrict the discussion to the one-dimensional case.

II. BROWNIAN CASE: KLEIN-KRAMERS, FOKKER-PLANCK, AND RAYLEIGH EQUATIONS

In the Brownian case, the Chapman-Kolmogorov equation (1), together with the definition (2) of the belonging transfer kernel, is readily integrated with respect to the position increment Δx. The integration over the velocity changes Δv is possible after a Taylor expansion of both the pdf W and the kernel ψ in powers of Δv so that the final result reads [12]

$$W(x,v,t)+\Delta t\frac{\partial W}{\partial t}=\left(1-\frac{\partial}{\partial v}\langle\Delta v\rangle+\frac{\langle(\Delta v)^2\rangle}{2}\frac{\partial^2}{\partial v^2}\right)W(x,v,t). \tag{7}$$

The necessary information about the velocity increments' moments, $\langle\Delta v\rangle$ and $\langle(\Delta v)^2\rangle$ is obtained from the stochastic Langevin equation [43]

$$m\frac{d^2x}{dt^2}=-\eta m\frac{dx}{dt}+F(x)+m\Gamma(t), \tag{8}$$

where $\Gamma(t)$ denotes a δ-correlated Gaussian noise. Accordingly, one finds [12]

$$\langle\Delta v\rangle=-\left(\eta v+\frac{F(x)}{m}\right)\Delta t,\quad\langle(\Delta v)^2\rangle=\frac{k_BT\eta}{m}\Delta t, \tag{9}$$

i.e., both moments are proportional to Δt [44].

Combining Eq. (7) with Eq. (9), one arrives at the kinetic equation for the pdf $W(x,v,t)$, the deterministic Klein-Kramers equation [4,12,45,46]

$$\frac{\partial W}{\partial t}=\left[-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}\left(\eta v-\frac{F(x)}{m}\right)+\frac{\eta k_BT}{m}\frac{\partial^2}{\partial v^2}\right]W(x,v,t). \tag{10}$$

Here and in the following, we make use of the Einstein relation $K=k_BT/(m\eta)$, connecting the friction and diffusion constants η and K [15]. Equation (10) is a bivariate Fokker-Planck equation describing the motion of a passive Brownian test particle of mass m under the influence of an external force field $F(x)$ in phase (position-velocity) space. On the right-hand side of Eq. (10), the first term describes the spatial drift due to the velocity of the test particle, the second term accounts for the friction and external force feedback to the velocity as expressed through the corresponding Langevin equation, and the third term represents the velocity diffusion, i.e., the spreading of the pdf $W(x,v,t)$ on the (x,v) field in the course of time.

The distribution in velocity space, related to Eq. (10) and without the external potential, is governed by the Rayleigh equation [3,47,48]

$$\frac{\partial W}{\partial t}=\eta\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(v,t) \tag{11}$$

with the corresponding Langevin equation $dv/dt=-\eta v+\Gamma(t)$. The Rayleigh equation controls the diffusion of the test particle in velocity space that is confined by the Ornstein-Uhlenbeck term $\eta(\partial/\partial v)vW(x,v,t)$ corresponding to the velocity damping term $-\eta v$ in the Langevin equation. Equation (11) thus describes the relaxation of the pdf $W(v,t)$ toward the stationary Maxwell distribution $W_{\rm st}(v)$, Eq. (5), with $E=mv^2/2$ and $N=\sqrt{\beta m/2\pi}$.

In the high-friction limit, one may neglect the inertial term in the corresponding stochastic differential equation (8), to obtain $dx/dt=F(x)/m\eta+(1/\eta)\Gamma(t)$, or the monovariate Fokker-Planck equation, often referred to as the Smoluchowski equation, [3,4,7,49,50]

$$\frac{\partial W}{\partial t}=\frac{1}{m\eta}\left(-\frac{\partial}{\partial x}F(x)+k_BT\frac{\partial^2}{\partial x^2}\right)W(x,t). \tag{12}$$

Equation (12) determines the diffusion of the test particle in position space under the influence of the external force field $F(x)$. Formally, the stationary solution $W_{\mathrm{st}}(x)$ of the Fokker-Planck equation (12) given by Eq. (5) for $E=V(x)$ can be obtained from the equilibrium solution $W_{\mathrm{st}}(x,v)$ of the Klein-Kramers equation (10) by integration over the velocity variable. However, in passing from the Klein-Kramers equation (10) itself to the Fokker-Planck equation (12), the additional telegrapher's term $(1/\eta)(\partial^2 W/\partial t^2)$ occurs, which can only be neglected in the long-time, high-friction limit $t \gg \eta^{-1}$ [51].

III. GENERALIZED CHAPMAN-KOLMOGOROV EQUATION

Let us now turn to the generalized Chapman-Kolmogorov equation (6). As mentioned before, two crucial ingredients for further study are specific choices for the waiting time pdf $w(t)$, and for the transfer kernel Ψ. For our purposes, we concentrate on such kernels that acquire the general form

$$\Psi(x-\Delta x,v-\Delta v;\Delta x,\Delta v)=\psi(v-\Delta v;\Delta v)p(\Delta x|t)w(t), \tag{13}$$

where the *a priori* conditional probability $p(\Delta x|t)$ connects the position increment Δx with the elapsed waiting time t. According to the continuous time random walk theory as developed in Ref. [42],[1] the waiting time pdf and consequently $p(\Delta x|t)$ as well, enter in the following convolution fashion:

$$W(x,v,t)=\int_0^t dt'\int_{-\infty}^{\infty}d(\Delta x)\int_{-\infty}^{\infty}d(\Delta v)W(x-\Delta x,v-\Delta v,t')\psi(v-\Delta v;\Delta v)p(\Delta x|t-t')w(t-t')+\phi(t)\delta(x)\delta(v). \tag{14}$$

For the underlying random walk process, this implies that the walker gets stuck in a certain position for a time span drawn from the waiting time pdf (Sec. V), or locks onto a given velocity mode for a random time determined by $w(t)$ (Secs. VI and VII). Moreover, we chose the initial condition to be $W_0(x,v)=\delta(x)\delta(v)$.

In general, the generalized Chapman-Kolmogorov equation describes a non-Markovian process due to the presence of the time convolution, a typical manifestation of memory [52,53]. The Brownian limit is nevertheless contained in Eq. (6) through the choice of the sharply peaked waiting time pdf

$$w(t)=\delta(t-\Delta t), \tag{15}$$

in connection with the conditional probability

$$p(\Delta x|t)=\delta(\Delta x-vt) \tag{16}$$

so that $w(t)p(\Delta x|t)=\delta(t-\Delta t)\delta(\Delta x-v\Delta t)$. Indeed, if one only considers the long-time limit, any narrow waiting time distribution possessing a finite characteristic waiting time

$$T\equiv\int_0^{\infty}tw(t)dt \tag{17}$$

leads back to the Brownian description.

The connection to the generalized master equation and the continuous time random walk model

Integration of the generalized Chapman-Kolmogorov equation (6) over velocity leads, after changing the dummy variable, to

$$W(x,t)=\int_0^t dt'\int_{-\infty}^{\infty}dx'\,W(x';x-x',t')\times\Psi(x';x-x')w(t-t')+\phi(t)W_0(x), \tag{18}$$

which is equivalent to the generalized master equation

$$\frac{\partial W(x,t)}{\partial t}=\int_0^t dt'\int_{-\infty}^{\infty}dx'\,W(x',t')K(x-x',x';t-t') \tag{19}$$

with the kernel $K(x-x',x';t-t')$, which depends on both the distance $x-x'$ and the departure site x'. Equivalently, we can write $\mathfrak{K}(x,x';t-t')\equiv K(x-x',x';t-t')$, $\mathfrak{K}$ being defined in terms of

$$\mathfrak{K}(x,x';u)=uw(u)\frac{\Psi(x,x')-\delta(x-x')}{1-w(u)}. \tag{20}$$

In Ref. [37], we demonstrated that the choice $\Psi(x,x')\equiv\lambda(x-x')[A(x')\Theta(x-x')+B(x')\Theta(x'-x)]$, where λ is the jump length distribution and the coefficients A and B are the local weights for going right or left, leads to the fractional Fokker-Planck equation (50) discussed below. Moreover, in the isotropic limit $A(x)=B(x)$, Eq. (19) corresponds to the standard continuous time random walk model with $\Psi(x,x')\equiv\lambda(x-x')$ [41,42].

[1] In essence, our notation $p(\Delta x|t)w(t)$ is equivalent to the continuous time random walk notation $\psi(x,t)$ for the jump pdf.

IV. FRACTIONAL RAYLEIGH EQUATION: A GENERALIZED CHAPMAN-KOLMOGOROV PROCESS WITH A LONG-TAILED WAITING TIME DISTRIBUTION

Under the condition that the position increments Δx are not distributed in a pathological way, the position average of Eq. (14) can be performed to result in the monovariate generalized Chapman-Kolmogorov equation

$$W(v,t)=\int_0^t dt'\int_{-\infty}^{\infty}d(\Delta v)W(v-\Delta v,t') \times\psi(v-\Delta v;\Delta v)w(t-t')+\phi(t)\delta(v) \quad (21)$$

for the pdf $W(v,t)$. The aforementioned Taylor expansion in powers of Δv up to second order, and evaluation of the belonging integral leads to

$$W(v,t)=\int_0^t dt'\left(1-\frac{\partial}{\partial v}\langle\Delta v\rangle+\frac{\langle(\Delta v)^2\rangle}{2}\frac{\partial^2}{\partial v^2}\right) \times W(v,t')w(t-t')+\phi(t)\delta(v). \quad (22)$$

Let us now consider a waiting time pdf of the one-sided Lévy stable type, $w(t)=L_\alpha^+(t/\tau)$ [18,54], or, more general, of the long-tailed form

$$w(t)\sim\frac{A_\alpha}{t^{1+\alpha}},\quad 0<\alpha<1, \quad (23)$$

which corresponds to a divergence of the characteristic waiting time T due to the ‘‘relatively frequent occurrence of very long waiting times.’’ For the appropriate choice of the constant A_α, the pdf $w(t)$ thus displays the following asymptotic behavior in Laplace space, $w(u)\equiv\int_0^t w(t)e^{-ut}dt$:

$$w(u)\sim 1-(u\tau)^\alpha. \quad (24)$$

We will call τ, or τ^α, the internal waiting time scale, i.e., we will regard it as the scaling unit of the waiting time process.

A further ingredient we need to evaluate in Eq. (22) is the special form of the moments of Δv. Let us assume for the time being that, in the force-free limit, they are given through

$$\langle\Delta v\rangle=-\eta v\tau^*,\quad \langle(\Delta v)^2\rangle=\frac{2k_BT\eta}{m}\tau^*, \quad (25)$$

with the ‘‘interaction time’’ constant τ^*, on the meaning of which we will comment below.

Putting together Eqs. (22), (24), and (25), we arrive at the Laplace space equation

$$W(v,u)=\left(1+\frac{\partial}{\partial v}\eta v\tau^*+\frac{k_BT\eta}{m}\tau^*\frac{\partial^2}{\partial v^2}\right)W(v,u)w(u) +\phi(u)\delta(v). \quad (26)$$

Noting that $\phi(u)=[1-w(u)]/u$ and neglecting terms of order $O(\tau^{2\alpha})$, we obtain the equation

$$W(v,u)-\frac{W_0(v)}{u}=u^{-\alpha}\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)\eta^* W(v,u), \quad (27)$$

with

$$\eta^*\equiv\eta\frac{\tau^*}{\tau^\alpha},\quad [\eta^*]=\sec^{-\alpha}. \quad (28)$$

The Laplace inversion of Eq. (27) whose right-hand side contains the form $u^{-\alpha}W(v,u)$ can be performed recalling the definition of a fractional integral according to Riemann and Liouville [55],

$${}_0D_t^{-\alpha}W(v,t)\equiv\frac{1}{\Gamma(\alpha)}\int_0^t dt'\frac{W(v,t')}{(t-t')^{1-\alpha}}, \quad (29)$$

which possesses the important property

$$\mathcal{L}\{{}_0D_t^{-\alpha}W(v,t)\}\equiv\int_0^\infty dt e^{-ut}{}_0D_t^{-\alpha}W(v,t)=u^{-\alpha}W(v,u). \quad (30)$$

Thus, we recover the convolution integral equation

$$W(v,t)-W_0(v)={}_0D_t^{-\alpha}\eta^*\left(\frac{\partial}{\partial v}+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(v,t), \quad (31)$$

which by definition of the fractional differential operator

$${}_0D_t^{1-\alpha}\equiv\frac{\partial}{\partial t}{}_0D_t^{-\alpha} \quad (32)$$

can be recast into the fractional Rayleigh equation [39,40]

$$\frac{\partial W}{\partial t}={}_0D_t^{1-\alpha}\eta^*\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(v,t). \quad (33)$$

Note that this is the fractional generalization of the Ornstein-Uhlenbeck process.

Discussion of the fractional Rayleigh equation

Let us first point out that the limit $\alpha\to 1$ of Eq. (33) corresponds to the Brownian Rayleigh equation [Eq. (11)]. This relates to the fact that for the Lévy stable form $w(t)=L_\alpha^+(t/\tau)$, this very limit $\alpha\to 1$ leads back to the δ form (15) with characteristic waiting time $T=\tau=\Delta t$. In the long-time limit, the same holds true for the Poissonian form $w(t)=\tau^{-1}e^{-t/\tau}$, or any waiting time pdf with finite T.

The velocity moments belonging to the fractional Rayleigh equation can be calculated directly from Eq. (33) by integration. Thus, for $\int_{-\infty}^{\infty}v dv\cdot$, one obtains the fractional relaxation equation

$$\frac{d}{dt}\langle v(t)\rangle+{}_0D_t^{1-\alpha}\eta^*\langle v(t)\rangle=0, \quad (34)$$

the Laplace transform of which (see Oldham and Spanier [55]) reveals [56]

$$\langle v(u)\rangle = \frac{v_0}{u+\eta^* u^{1-\alpha}}, \tag{35}$$

which in turn defines the Mittag-Leffler function [57,58]

$$\langle v(t)\rangle = v_0 E_\alpha(-\eta^* t^\alpha) \equiv \sum_{n=0}^{\infty} \frac{(-\eta^* t^\alpha)^n}{\Gamma(1+\alpha n)}. \tag{36}$$

The Mittag-Leffler function is thus a natural generalization of the exponential function. For $0<\alpha<1$ it is strictly monotonic, and interpolates between the initial stretched exponential pattern

$$\langle v(t)\rangle \sim v_0 \exp\left(-\frac{\eta^* t^\alpha}{\Gamma(1+\alpha)}\right) \tag{37}$$

and the final inverse power-law decay

$$\langle v(t)\rangle \sim [\eta^* t^\alpha \Gamma(1-\alpha)]^{-1}. \tag{38}$$

The second velocity moment of the fractional Rayleigh process shows the Mittag-Leffler equilibration

$$\langle v(t)^2\rangle = v_0^2 E_\alpha(-2\eta_\alpha t^\alpha) + \frac{k_B T}{m}[1-E_\alpha(-2\eta_\alpha t^\alpha)], \tag{39}$$

toward the stationary thermal value $\langle v^2\rangle_{\mathrm{th}} \equiv k_B T/m$.

Introducing the separation ansatz $W_n(v,t) = T_n(t)\varphi_n(v)$ for the eigenvalue λ_n of index n [27,35,36], the velocity eigenequation is equivalent to the one obtained in the Brownian case, see, e.g., Ref. [4], and the temporal eigenequation corresponds to the fractional relaxation equation (34) that determines the Mittag-Leffler relaxation

$$T_n(t) = E_\alpha(-\lambda_n t^\alpha) \tag{40}$$

for the relaxation of the mode n with eigenvalue λ_n.

The observed Mittag-Leffler relaxation of moments and modes is closely related to the Laplace space rescaling

$$W_\alpha(v,u) = \frac{\eta}{\eta^*} u^{\alpha-1} W_1\left(v, \frac{\eta}{\eta^*} u^\alpha\right) \tag{41}$$

fulfilled by the solutions of the fractional Rayleigh equation (33) labeled $W_\alpha(v,u)$, and the solution of the Brownian Rayleigh equation (11), $W_1(v,u)$. Consequently, the fractional solution is positive [40], and the exponential relaxation so typical for Brownian processes turns into the Mittag-Leffler pattern that can easily be seen from the analogous rescaling of the velocity moment,

$$\mathcal{L}\{v_0 e^{-\eta t}\} = \frac{v_0/\eta}{1+u/\eta} \overset{\text{rescaling}}{\to} \frac{v_0}{u+\eta^* u^{1-\alpha}} \tag{42}$$

and comparison with the definition (35).

Concluding this section, we remark that *all* processes based upon the generalized Chapman-Kolmogorov equation (14), governed by a broad waiting time pdf of the type (23), with diverging characteristic time, lead to a Mittag-Leffler equilibration toward the Maxwell distribution, irrespectively of the conditional probability $p(\Delta x|t)$. In that sense, the Mittag-Leffler relaxation acquires a universal character in anomalous transport of such types that feature a scale-free, self-similar waiting time pdf.

V. "LÉVY SNEAKING": SLOW TRANSPORT PROCESS GOVERNED BY TRAPPING

Let us now include the position space dynamics into the discussion. We start off with the consideration of a process whose force-free limit describes subdiffusion, Eq. (4) with $0<\alpha<1$. This process corresponds to a multiple trapping scenario where the test particle moves Brownian style in accordance to the Langevin equation with white Gaussian noise, Eq. (8), and gets successively immobilized in traps whose mean distance is $\Delta = \langle v^2\rangle_{th}\tau^*$ [39]. As it is supposed that the trapping is strong, the time spans spent in the immobilized state are ruled by the stable waiting time pdf (23).

As shown in Ref. [39], this process can be modeled by the conditional probability

$$p(\Delta x|t) \equiv p(\Delta x) = \delta(\Delta x - v\tau^*), \tag{43}$$

which thus leads to a decoupled formulation in the sense that p does not involve the waiting time explicitly, as it does in the processes discussed in Secs. VI and VII. Only for short times, $p = \delta(\Delta x - vt)$. It is shown in Ref. [39] that this initial behavior can be neglected in the long-time limit.

With a small error only in concern of this long-time limit, the governing generalized Chapman-Kolmogorov equation of this process takes on the form [39]

$$W(x,v,t) = \int_0^t dt' \int_{-\infty}^{\infty} d(\Delta v) W(x - v\tau^*, v - \Delta v, t') \times \psi(v-\Delta v; \Delta v) w(t-t') + \phi(t)\delta(x)\delta(v). \tag{44}$$

The velocity increment integration can be performed once the corresponding means are determined. As the trapping events are (kinetic energy-preserving) interruptions of the Langevin controlled process (8), these moments are given through

$$\langle \Delta v\rangle = -\eta v\tau^* + \frac{F(x)}{m}\tau^*; \quad \langle (\Delta v)^2\rangle = 2\frac{k_B T\eta}{m}\tau^*, \tag{45}$$

where τ^*, the mean time between successive trapping events, enters linearly in both expressions. Readily, the integration over Δv leads to

$$W(x,v,u)=\left(1-v\tau^*\frac{\partial}{\partial x}-\frac{F(x)}{m}\tau^*\frac{\partial}{\partial v}+\frac{\partial}{\partial v}\eta v\tau^*+\frac{k_BT\eta}{m}\tau^*\frac{\partial^2}{\partial v^2}\right)W(x,v,u)w(u)+\frac{1-w(u)}{u}W_0(x,v). \quad (46)$$

Reshuffling terms and inverting to t analogously to the steps in the preceding Section, one arrives at the fractional Klein-Kramers equation (FKKE)

$$\frac{\partial W}{\partial t}={}_0D_t^{1-\alpha}\left[-v^*\frac{\partial}{\partial x}+\frac{\partial}{\partial v}\left(\eta^*v-\frac{F^*(x)}{m}\right)+\frac{\eta^*k_BT}{m}\frac{\partial^2}{\partial v^2}\right]W(x,v,t), \quad (47)$$

where $v^*\equiv v\tau^*/\tau^\alpha$ with $[v^*]=\mathrm{cm\,sec}^{-\alpha}$, $\eta^*\equiv\eta\tau^*/\tau^\alpha$ with $[\eta^*]=\mathrm{sec}^{-\alpha}$, and $F^*(x)\equiv F(x)\tau^*/\tau^\alpha$ with $[F^*]=\mathrm{cm\,g\,sec}^{-1-\alpha}$. Note that the Stokes operator $(\partial/\partial t+v\partial/\partial x)$ from the standard Klein-Kramers equation (10) [12] is replaced by the operator $(\partial/\partial t+{}_0D_t^{1-\alpha}v^*\partial/\partial x)$ that shows the nonlocal drift response.

A. Discussion of the fractional Klein-Kramers equation describing ''Lévy sneaking''

In the FKKE (47), the entire Klein-Kramers operator in the square brackets acts non-locally in time, i.e., drift, friction, and diffusion terms are under the time convolution and thus affected by the memory. Consequently, the (x,v)-averaged position $\langle\langle x(t)\rangle\rangle$ is related to its velocity counterpart through the non-Newtonian relation

$$\frac{d}{dt}\langle\langle x(t)\rangle\rangle=\frac{\tau^*}{\tau^\alpha}{}_0D_t^{1-\alpha}\langle\langle v(t)\rangle\rangle, \quad (48)$$

which seems to contradict the Brownian relation $(d/dt)\langle\langle x(t)\rangle\rangle=\langle\langle v(t)\rangle\rangle$. This ''violation'' results entirely from the camouflaging effect of the introduction of the long-tailed waiting time pdf $w(t)$. Indeed, in the underlying Langevin equation (8) governing the nontrapping regimes of the Lévy sneaking process, the noise average is in full accordance with Newton's laws: $m(d^2/dt^2)\langle x\rangle_\Gamma=-\eta\langle v\rangle_\Gamma+F(x)$. It is solely the averaging over the sequence of trapping and nontrapping events plus the dominance of the trapping regimes due to the long-tailed waiting time pdf (23) that brings about the behavior (48).

Moreover, the FKKE (47) is separable in the sense that a separation ansatz decouples the equation into a temporal and a spatial eigensolution so that the fractional Klein-Kramers mode relaxation in the Lévy sneaking problem follows the Mittag-Leffler pattern [39].

As the multiple trapping process described by the FKKE (47) is assumed to be kinetic energy conserving, single relaxation events $-\eta v\tau^*$ due to friction, i.e., due to the effective interaction of the particle with its environment, are distributed such that the separating time intervals are distributed according to the waiting time pdf (23). Therefore, the equilibration of the velocity pdf $W(v,t)$ of the associated force-free process is described by the fractional Rayleigh equation (33) derived in the preceding section, as it should be.

B. The associated fractional Fokker-Planck equation of the ''Lévy sneaker''

Let us now turn toward the overdamped limit of the FKKE (47). Using the same steps as in the Brownian case (see Ref. [51]), the integration of the FKKE (47) over velocity $\int_{-\infty}^{\infty}dv\cdot$, and v times Eq. (47) over velocity $\int_{-\infty}^{\infty}v\,dv\cdot$ leads to two independent equations, the combination of which produces the kinetic equation

$$\frac{\partial W}{\partial t}+{}_0D_t^{1+\alpha}\frac{1}{\eta^*}W={}_0D_t^{1-\alpha}\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta_\alpha}+K_\alpha\frac{\partial^2}{\partial x^2}\right)W(x,t) \quad (49)$$

from which, in the high-friction limit, one is led to the fractional Fokker-Planck equation (FFPE)

$$\frac{\partial W}{\partial t}={}_0D_t^{1-\alpha}\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta_\alpha}+K_\alpha\frac{\partial^2}{\partial x^2}\right)W(x,t), \quad (50)$$

which was discussed in detail in Ref. [36], and was derived from a generalized master equation and a nonhomogeneous random walk, in Refs. [37,38]. The constants η_α and K_α introduced in this derivation are now defined as

$$\eta_\alpha\equiv\frac{\eta\tau^\alpha}{\tau^*}^{-1},\ \ [\eta_\alpha]=\mathrm{sec}^{\alpha-2};$$

$$K_\alpha\equiv\frac{k_BT}{m\eta_\alpha},\quad [K_\alpha]=\mathrm{cm}^2\ \mathrm{sec}^{-\alpha}. \quad (51)$$

These relations show that the generalized coefficients are based on the proper dynamical quantities η, m, and τ^* and that the fractional dimensions emanate from the rescaling with τ^α, or, in other words, through the introduction of the fractal waiting time distribution, Eq. (23). Moreover, the generalized Einstein-Stokes [59] relation connecting K_α with η_α now follows directly from the derivation.

As pointed out in Ref. [36], the fractional solution of the FFPE (50) can be connected to its Brownian counterpart through the scaling relation

$$W_\alpha(x,u)=\frac{\eta_\alpha}{\eta}u^{\alpha-1}W_1\left(x,\frac{\eta_\alpha}{\eta}u^\alpha\right), \quad (52)$$

which is equivalent to Eq. (41) belonging to the fractional Ornstein-Uhlenbeck process, in turn formally a special case of the FFPE (50). Accordingly, the mode relaxation of the FFPE (50) is given through the same Mittag-Leffler pattern $T_n(t)=E_\alpha(-\lambda_\alpha t^\alpha)$ as derived for the corresponding FKKE (47).

The force-free mean-squared displacement that corresponds to the FFPE (50) reads

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha, \tag{53}$$

which is equal to Eq. (4) with $K_\alpha^* \equiv K_\alpha/\Gamma(1+\alpha)$.

VI. LÉVY WALKS AND THEIR GENERALIZATION: ENHANCED TRANSPORT WITH LÉVY TYPE TRAJECTORIES

In this section, we investigate the relation of the generalized Chapman-Kolmogorov framework to the Lévy walk model that is defined in continuous time random walk theory [42]. According to the latter, a Lévy walk is governed by the jump pdf $\psi(x,t) = w(t)p(x|t)$ that determines the step length and waiting time for the belonging random walk process. The conditional probability $p(x|t)$ introduces a time cost that penalizes long jumps by a high time cost. This is the crucial difference that sets the Lévy walk process apart from the Markovian Lévy flight. The former exhibits a finite mean-squared displacement and thus delivers a physically sensible theory for a massive test particle whereas the latter possesses a diverging second moment. It is of interest to show the connection of our generalized Chapman-Kolmogorov equation to the Lévy walk, as the phase-space formulation, in principle, allows for an extension of this Lévy walk model for nontrivial velocity distributions and in the presence of external force fields. Here we concentrate on such Lévy walks for which the spatiotemporal coupling is linear, i.e., where $p(x|t) = \delta(x - vt)$. Note that the following reasoning can be extended to the general coupling described in Ref. [42], or beyond.

Starting off with the generalized Chapman-Kolmogorov equation (14), we choose the δ coupling

$$p(\Delta x|t) \equiv \frac{1}{2}\delta(|x| - v_0 t), \tag{54}$$

where the absolute value is necessary as we additionally require the sharp velocity distribution

$$\psi(v - \Delta v; \Delta v) \equiv \delta(\Delta v - v_0). \tag{55}$$

Namely, if we choose

$$W_0(x,v) = \delta(x)\,\delta(v - v_0), \tag{56}$$

nominally no changes in the apparent velocity of the test particle occur, which has now to be expressed through the choice (54).

Then, the integration over the velocity increments Δv and over the velocity v reveals

$$W(x,t) = \int_0^t dt' \int_{-\infty}^{\infty} d(\Delta x) W(x - \Delta x, t') \times \frac{1}{2}\delta(|\Delta x| - v_0(t-t'))w(t-t') + \phi(t)\,\delta(x), \tag{57}$$

which by change of convolution variables and relabeling of the dummy variable Δx can be recast to

$$W(x,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' W(x',t')\,\psi(x - x', t - t') + \phi(t)\,\delta(x) \tag{58}$$

with the jump pdf

$$\psi(x,t) = \frac{1}{2}\delta(|x| - v_0 t)w(t), \tag{59}$$

which is exactly equivalent to the Lévy walk definition in the jump picture given in Ref. [42].

For the waiting time pdf with finite characteristic time T but infinite second moment, such as

$$w(t) \sim \frac{A_\beta}{t^{1+\beta}}, \quad 1 < \beta < 2, \tag{60}$$

it can be shown that the associated mean-squared displacement follows [42]

$$\langle x^2(t)\rangle \propto t^{3-\beta}, \tag{61}$$

which describes subballistic superdiffusion and is not to be confused with the scaling results discussed in the connection with Lévy flights [60].

A striking quality of Lévy walks is that their trajectory resembles a veritable Lévy flight, i.e., a process where each jump length is distributed in Lévy stable fashion, with a diverging second moment. This similarity stems from the fact that for increasing time the allowed jump length window in the Lévy walk model grows, i.e., the penalizing time consumption for long jumps becomes less relevant. The same phenomenon is observed for the propagator whose wings more and more approach a true Lévy pdf, except for the cutoff δ spikes that correspond to particles locked onto the v_0 mode [42].

The calculation of a kinetic equation corresponding to the Lévy walk Chapman-Kolmogorov equation (58) is not as straightforward as in the case of Lévy sneaking, exactly due to the possibly very long sojourns coming into play, and creating the approaching to a long-tailed pdf. As shown below, this forbids a truncation in the belonging Fourier space expansion that is always taken in the continuous time random walk, but only for the calculation of the first two moments and not for the associated pdf. Otherwise, all terms of the Fourier expansion have to be carried along, in a Kramers-Moyal type expansion.

VII. ''LÉVY RAMBLING'': SUBBALLISTIC ROVING IN THE SMALL WAVE-NUMBER LIMIT

In this section, we introduce a new process whose definition exactly draws this small wave-number limit in the Fourier representation that is not commensurate with Lévy walks (see above). In this process, the particle is assumed to stay fixed in a given propagation mode, but the length of the belonging excursions is truncated. The associated process is interesting in its own way, and we will argue that it might prove to be an interesting variant for the description of subballistic, enhanced transport in an external force field, and with finite moments. Indeed, this process is closer to the wave equation than to the diffusion equation, as we will show. The force-free position space analog of this process has been investigated in Ref. [61], where its physical relevance was noted by analogy to results obtained for Richardson diffusion.

The basic definition of ''Lévy rambling'' is very similar to the Lévy walk, i.e., we have an explicit spatiotemporal coupling of the form (54). Moreover, we define the moments of the velocity increments through

$$\langle \Delta v\rangle = -\eta v\tau^* + \frac{F(x)}{m}t;\quad \langle(\Delta v)^2\rangle = 2\frac{k_BT\eta}{m}\tau^*. \tag{62}$$

By this definition, we assume that the influence of the friction and the entropic parts enter through the effective ''interaction time'' scale τ^* whereas the force constantly acts upon the test particle. Choosing a long-tailed waiting time distribution with a diverging mean-squared displacement, the finite τ^* becomes negligibly small and can be interpreted as a pointlike interaction. Comparing to the physical picture of a collision model, these assumptions are physically sensible [62]. The final and crucial definition, the small wave-number limit, is introduced below.

Putting all the ingredients together, we arrive at the generalized Chapman-Kolmogorov equation

$$W(x,v,t) = \int_0^t \left(1 + \frac{\partial}{\partial v}\eta v\tau^* - \frac{\partial}{\partial v}\frac{F(x)}{m}(t-t') + \frac{k_BT\eta}{m}\tau^*\frac{\partial^2}{\partial v^2}\right) \times W(x-v(t-t'),v,t')w(t-t') + \phi(t)\delta(x)\delta(v), \tag{63}$$

where the operator $\partial/\partial v$ acts solely on the second argument of $W(x-v(t-t'),v,t')$. For the further evaluation we introduce a Fourier transformation, recovering

$$W(k,v,t) = \int_0^t dt' \left(1 + \frac{\partial}{\partial v}\eta v\tau^* - \frac{\partial}{\partial v}\frac{F}{m}(t-t') + \frac{k_BT\eta}{m}\tau^*\frac{\partial^2}{\partial v^2}\right) W(k,v,t')e^{ikv(t-t')}w(t-t') + \phi(t)\delta(v), \tag{64}$$

where we note that the force F should actually be written in a Fourier convolution with W. We prefer to suppress the explicit occurrence of this, however. Note that in the last step we made use of the translation theorem $f(x-a)\to e^{ika}f(k)$ of the Fourier transformation. With the equivalent theorem for the Laplace transformation, we arrive at the equation

$$W(k,v,u) = \left(1 + \frac{\partial}{\partial v}\eta v\tau^* + \frac{\partial^2}{\partial v\,\partial u}\frac{F}{m} + \frac{k_BT\eta}{m}\tau^*\frac{\partial^2}{\partial v^2}\right) W(k,v,u)w(u-ikv) + \phi(u)\delta(v), \tag{65}$$

where $\partial/\partial u$ solely acts upon the function $w(u-ikv)$, and we employ the theorem $tg(t)\to(d/du)g(u)$.

It is at this point where we introduce the small wave-number limit, namely, through the approximation

$$\begin{aligned} w(u-ikv) &\sim 1-(u\tau)^\alpha\big|_{u-ikv} \\ &= 1-(u\tau)^\alpha\left(1-\frac{ikv}{u}\right)^\alpha \sim 1-(u\tau)^\alpha\left(1-\alpha\frac{ikv}{u}\right) \\ &\sim w(u)+\alpha\tau^\alpha ikvu^{\alpha-1}, \end{aligned} \tag{66}$$

which we terminate after the first order in k. In addition, we have to specify which terms we will take along in our long-time, small wave-number (long-wave) limit. Multiplying expression (66) with the operators in the round brackets in Eq. (65), we take both terms in Eq. (66) along with the ''1,'' whereas all the remaining terms in the round brackets are only combined with $w(u)$, due to the observation that the neglected terms are of higher order concerning the combination of the wave number k and the corresponding Fourier quantity for the velocity.[2]

Observing that

$$\frac{d}{du}w(u) = -\alpha\tau^\alpha u^{\alpha-1} + O(\tau^{2\alpha}), \tag{67}$$

Eq. (65) is recast into

[2]Note that this rule results from the typical diffusion limit $k\to 0, u\to 0$ in Fourier-Laplace space [42].

$$W(k,v,u)=\left(w(u)+\alpha\tau^{\alpha}ikvu^{\alpha-1}+w(u)\frac{\partial}{\partial v}\eta v\tau^{*}+\alpha\tau^{\alpha}u^{\alpha-1}\frac{F}{m}\frac{\partial}{\partial v}+w(u)\frac{k_BT\eta\tau^{*}}{m}\frac{\partial^2}{\partial v^2}\right)\times W(k,v,u)+\frac{1-w(u)}{u}\delta(v). \tag{68}$$

With the usual differentiation theorem $ikf(k)\to -(d/dx)f(x)$, we find the integral form of the Lévy rambling fractional Klein-Kramers equation

$$W(x,v,t)-W_0(x,v)+\alpha\,{}_0D_t^{-1}\left(v\frac{\partial}{\partial x}+\frac{F(x)}{m}\frac{\partial}{\partial v}\right)W(x,v,t)$$
$$={}_0D_t^{-\alpha}\eta^{*}\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(x,v,t) \tag{69}$$

through Fourier and Laplace inversion.

Differentiation in respect to t leads to the differential form

$$\frac{\partial W}{\partial t}+\alpha v\frac{\partial W}{\partial x}+\alpha\frac{F(x)}{m}\frac{\partial W}{\partial v}$$
$$={}_0D_t^{1-\alpha}\eta^{*}\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(x,v,t). \tag{70}$$

Through a possible rescaling of the position variable ($x\to x/\alpha$) and the force ($F\to\alpha F$), we arrive at the final form of the Lévy rambling FKKE,

$$\frac{\partial W}{\partial t}+v\frac{\partial W}{\partial x}+\frac{F(x)}{m}\frac{\partial W}{\partial v}$$
$$={}_0D_t^{1-\alpha}\eta^{*}\left(\frac{\partial}{\partial v}v+\frac{k_BT}{m}\frac{\partial^2}{\partial v^2}\right)W(x,v,t), \tag{71}$$

which is equivalent to the equation introduced by Barkai and Silbey in Ref. [40].

A. Discussion of the fractional Klein-Kramers equation (71) describing ''Lévy rambling''

The stationary solution of Eq. (71) being defined via $\partial W/\partial t=0$, is obtained by requiring both expressions

$$v\,\partial W/\partial x+[F(x)/m](\partial W/\partial v)$$

and

$$[\partial/\partial v\, v+(k_BT/m)(\partial^2/\partial v^2)]W(x,v,t)$$

to vanish simultaneously. Otherwise, due to the fractional differentiation ${}_0D_t^{1-\alpha}1\propto t^{\alpha-1}$ of a constant, one side would be explicitly time-dependent. With the product ansatz $W_{\mathrm{st}}(x,v)=X(x)V(v)$, we find the Gibbs-Boltzmann form (5).

The FKKE (71) thus governs the relaxation toward Gibbs-Boltzmann equilibrium. Accordingly, it differs from the non-equilibrium stationary solution obtained for Lévy flights [31,32], and which also characterize Lévy walks. Moreover, the pdf $W(x,v,t)$ decays exponentially so that the associated trajectory in position space is *not* of the Lévy type (compare to Ref. [63]).

Due to its derivation that distinguishes the drift terms on the left-hand side of Eq. (71) from the friction and diffusion terms on its right-hand side where the latter are under the fractional operator, the macroscopically averaged position is of Newtonian character, i.e.,

$$\frac{d}{dt}\langle\langle x(t)\rangle\rangle=\langle\langle v(t)\rangle\rangle. \tag{72}$$

This observation sets the FKKE (71) clearly apart from the FKKE (47). For the latter, the analogous relation (48) highlights the non-Newtonian character brought about by the camouflaging waiting time averaging in which the broadly distributed trapping times win out in the competition with the Langevin dominated nontrapping motion events, an individual of these events lasting for the finite average time τ^{*}. Here we encounter the different case that, according to the assumption that the test particle, never being trapped, constantly responds to the external force field $F(x)$ that is mirrored in the moments (62). Apparently, this property is preserved in the small wave-number limit (66).

Note that due to this different quality in the temporal evolution separating the phase spatial derivatives on either side of the FKKE (71), the associated process does not separate into the product of a purely temporal and a phase spatial part as it was observed in Sec. V.

B. The associated fractional Fokker-Planck equation of the ''Lévy rambler''

Applying Davies's recipe [51] of velocity integration to the FKKE (71), we obtain the fractional telegrapher's type equation

$$\frac{1}{\eta^{*}}\,{}_0D_t^{2+\alpha}W+\frac{\partial^2}{\partial t^2}W={}_0D_t^{\alpha}\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta^{*}}+K_{2-\alpha}\frac{\partial^2}{\partial x^2}\right)W(x,t) \tag{73}$$

with the generalized diffusion constant

$$K_{2-\alpha}\equiv\frac{k_BT}{m\eta^{*}},\quad [K_{2-\alpha}]=\mathrm{cm}^2\ \mathrm{sec}^{\alpha-2}. \tag{74}$$

For short times, this process is governed through the equation

$$\frac{1}{\eta^*}\frac{\partial^2}{\partial t^2}W=\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta^*}+K_{2-\alpha}\frac{\partial^2}{\partial x^2}\right)W(x,t), \quad (75)$$

highlighting the ballistic motion that is to be expected for times before velocity changes occur, and for which the particles thus all move in their given original direction. This is most obvious in the force-free limit in which Eq. (75) reduces to the wave equation

$$\frac{\partial^2}{\partial t^2}W=\tilde{v}\frac{\partial^2}{\partial x^2}W(x,t), \quad (76)$$

where $\tilde{v}\equiv\eta^* K_{2-\alpha}$. Note that such a behavior can also be obtained in the Lévy sneaking model described in Sec. V if one avoids drawing the long-time limit.

In the usual high-friction limit, one finds the fractional Fokker-Planck equation

$$\frac{\partial^2}{\partial t^2}W={}_0D_t^{\alpha}\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta^*}+K_{2-\alpha}\frac{\partial^2}{\partial x^2}\right)W(x,t). \quad (77)$$

Note the second-order time derivative on the left-hand side, indicating that it is essentially a subballistic motion. In this model, the generalized friction coefficients in both the FKKE (71) and the FFPE (77) are given by η^*. The force-free mean-squared displacement belonging to Eq. (77) is given through

$$\langle x^2(t)\rangle=\frac{2K_{2-\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha}, \quad (78)$$

which corresponds to Eq. (4) with $K^*\equiv K_{2-\alpha}/\Gamma(3-\alpha)$, and issues the typical Lévy walk behavior displayed in Eq. (61). This equivalence originates in the full validity of the small wave-number limit corresponding to the truncated representation of the Lévy walk propagator in Ref. [42] for the calculation of the mean-squared displacement.

Let us introduce the separation ansatz $W(x,t)=T(t)\varphi(x)$ into Eq. (77). This leads to the fractional equation

$$\frac{d^2T_n}{dt^2}=-\lambda_n\,{}_0D_t^{\alpha}T_n(t) \quad (79)$$

for the temporal eigensolution where λ_n denotes the nth eigenvalue. On taking the Laplace transformation of Eq. (79), one has to provide two initial values for which we choose $T(0)=1$ and $dT/dt|_{t=0}=0$. Consequently, we obtain

$$T_n(u)=\frac{1/u}{1+\lambda_n u^{\alpha-2}}. \quad (80)$$

This result can be back-transformed into the t domain, making use of the definition of the Mittag-Leffler function:

$$T_n(t)=E_{2-\alpha}(-\lambda_n t^{2-\alpha}) \quad (81)$$

from which the series expansion

$$T_n(t)=\sum_{j=0}^{\infty}\frac{(-\lambda_n t^{2-\alpha})^j}{\Gamma[1+(2-\alpha)j]} \quad (82)$$

can be inferred. Note that for the belonging Mittag-Leffler index $2-\alpha>1$ there is no theorem for the long-time asymptotics. Wiman showed [58,64] that there is an odd number of negative zeroes, which holds for our case. Special representations can be obtained for rational numbers, e.g., for $\alpha=1/2$ one gets

$$E_{3/2}(-\lambda_n t^{3/2})=\frac{\exp\left(\frac{3}{2}\lambda_n^{2/3}t\right)+2\cos\left(\frac{\sqrt{3}\lambda^{2/3}t}{2}\right)}{3\exp\left(\frac{\lambda_n^{2/3}t}{2}\right)}-\frac{4\lambda_n t^{3/2}{}_1F_3\left(1;\frac{5}{6},\frac{7}{6},\frac{3}{2};\frac{\lambda_n^2t^3}{27}\right)}{3\sqrt{\pi}} \quad (83)$$

and one clearly recognizes superimposed oscillations that are typical for any $\alpha\in[0,1)$ in Eq. (81).

The investigation of the FFPE (77) in the force-free limit leads to interesting consequences, as shown in Ref. [61]. In the presence of an external force, the behavior of Eq. (77) requires special attention, especially in respect to the positivity of the belonging pdf $W(x,t)$. This is topic of a forthcoming investigation [65].

VIII. LÉVY FLYING: RANDOM MOTION BEYOND FINITE MOMENTS

In this final section, we address the determination of the FKKE for a Lévy flight process. In the continuous time random walk framework such a process is defined in terms of a Poissonian waiting time pdf $w(t)=\tau^{-1}\exp(-t/\tau)$, or Eq. (15), and the jump length pdf is Lévy stable, i.e., the jump pdf is given through [18,54]

$$\psi(x,t)=L_{\mu}(x)\tau^{-1}e^{-t/\tau}\sim\frac{B_{\mu}}{|x|^{1+\mu}}\tau^{-1}e^{-t/\tau}, \quad (84)$$

where we choose $1<\mu<2$. In Fourier-Laplace space, the corresponding asymptotic limit reveals

$$\psi(k,u)\sim 1-u\tau-\sigma^{\mu}|k|^{\mu}. \quad (85)$$

As was discussed by Seshadri and West [28] and by Peseckis [29], and later in the context of random environments by Fogedby [31] and Honkonen [32], this concept leads to the FKKE

$$\frac{\partial W}{\partial t}+v\frac{\partial W}{\partial x}+\frac{F(x)}{m}\frac{\partial W}{\partial v}=\eta\left(\frac{\partial}{\partial v}v+K^{\mu}{}_{-\infty}D_x^{\mu}\right)W(x,v,t), \quad (86)$$

where the fractional Weyl operator is defined through [55]

$${}_{-\infty}D_x^{-\mu}W(x,v,t)=\frac{1}{\Gamma(\mu)}\int_{-\infty}^{x}dx'\frac{W(x',v,t)}{(x-x')^{1-\mu}}, \quad (87)$$

in the definition of which we include a phase factor so that it fulfills the generalized differentiation theorem

$$\mathcal{F}\{{}_{-\infty}D_x^{\mu}W(x,v,t)\}\equiv\int_{-\infty}^{\infty}dx e^{ikx}{}_{-\infty}D_x^{\mu}W(x,v,t)$$
$$=-|k|^{\mu}W(k,v,t). \tag{88}$$

Discussion of the Lévy flight Klein-Kramers equation and the associated Fokker-Planck equation

One obtains from the FKKE (86) in the usual way the fractional telegrapher's-type equation

$$\frac{\partial W}{\partial t}+\frac{1}{\eta}\frac{\partial^2 W}{\partial t^2}=\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta}+K^{\mu}{}_{-\infty}D_x^{\mu}\right)W(x,v,t) \tag{89}$$

via v integration. Neglecting the second-order time derivative in the high-friction limit, one recovers the fractional Fokker-Planck equation that was inferred in Refs. [29] for Lévy flights in a random environment, and discussed in further detail in Refs. [31,32]. This type of fractional Klein-Kramers equation and its related fractional Fokker-Planck equation, i.e., the Lévy noise approach, leads to the divergence of the mean-squared displacement $\langle x(t)^2\rangle=\infty$, as is typically found for Lévy flights [42,60], and it will not be pursued here further.

Note that the FKKE (86) is similar to an equation derived within a quantum-mechanical picture through a random matrix formalism by Kusnezov *et al.* [33].

IX. CONCLUSIONS

We have generalized a formerly proposed Chapman-Kolmogorov equation to continuous time phase-space dynamics, employing the Chandrasekhar notation. It has been demonstrated that this new equation is the common footing for a variety of stochastic models that describe normal and anomalous transport in external fields in the underdamped as well as overdamped limit.

Three crucial ingredients have to be supplied to the generalized Chapman-Kolmogorov equation in order to arrive at a certain model.

(i) The waiting time pdf $w(t)$ that determines the time spans between successive velocity changing events. This pdf $w(t)$ either describes the time of being trapped at a given site, or the time spent in a velocity mode, leading to either ''Lévy sneaking'' that corresponds to force-free subdiffusion, or to Lévy walking or ''rambling'' from which the force-free sub-ballistic motion is derived. In the Markovian cases of Lévy flights and Brownian motion, the waiting time pdf becomes obsolete and can be replaced by an average time step Δt.

(ii) The spatiotemporal coupling $p(\Delta x|t)$ relating the covered distance Δx per motion event with the corresponding waiting time *t*. Brownian motion, ''Lévy sneaking,'' and flying are modeled through an uncoupled form $\delta(\Delta x - v\tau^*)$, where τ^* is regarded as a small parameter and denotes an effective time scale in which the Langevin régime governs the motion of the test particle.

(iii) The moments of the velocity increments $\langle\Delta v\rangle$ and $\langle(\Delta v)^2\rangle$ that have to be supplied in order to perform the transition to the belonging (fractional) Klein-Kramers equation. These moments define the interaction of the test particle with its environment through friction, external forcing, and entropy. We have assumed throughout that these moments are based on a stochastic Langevin dynamics with δ-correlated Gaussian noise $\Gamma(t)$.

Apart from the classical Brownian case, ''Lévy sneaking'' is possibly the best founded case of the anomalous models discussed. Its fundamental ingredient, the trapping mechanism, has been recognized as the mechanism underlying the dispersive charge carrier transport in amorphous semiconductors [66,67], the motion of excess electrons in liquids [68], and it occurs in the phase-space dynamics of chaotic Hamiltonian systems [69].

Lévy flights are physically sensible only for a limited range of problems such as the diffusion in energy space encountered in single molecule spectroscopy [70]. Otherwise, massive particles are required to have a finite speed. This is especially crucial in the Lévy flight model where extremely long jumps are permitted to be performed instantaneously. Thus, Lévy flight modeling of test particles in real space can at most be an approximation; compare, however, Refs. [72,73]. Moreover, it might be questioned whether the Langevin equation description with Lévy noise can hold in respect to the linear friction assumption.

Two ways out of these problems were described. One is the well-known Lévy walk that can be generalized to external fields and phase space through the presented approach. The second is ''Lévy rambling'' that was derived as the small wave-number limit of Lévy walking. Both lead to finite moments of any order and include moving humps or spikes that lead to the oscillations superimposed to the relaxation behavior. Lévy rambling as well as the phase-space analog of Lévy walking are to be studied further in a future work.

The Brownian process, ''Lévy sneaking,'' and ''Lévy rambling'' are characterized through an equilibration toward the classical Gibbs-Boltzmann distributions. The related generalized Einstein relations as well as the validity of the second Einstein relation [36–40] are tightly related to this dynamical behavior close to classical thermal equilibrium.

A ubiquitous behavior for all processes governed by a self-similar waiting time pdf with diverging characteristic waiting time is the Mittag-Leffler velocity equilibration according to the fractional Rayleigh equation. In this case, the stationary solution corresponds to the Maxwell distribution.

It should finally be emphasized that for an appropriate choice of the transfer kernel, also mixed forms of the associated deterministic kinetic equations with nonordinary derivatives in both space and time can be recovered as they were, for instance, discussed in Refs. [37,71] on the fractional Fokker-Planck level.

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